

Langlands parameters for epipelagic representations of GL_n

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ABSTRACT. Let F be a non-Archimedean local field. An irreducible cuspidal representation of $\mathrm{GL}_n(F)$ is epipelagic if its Swan conductor equals 1. We give a full and explicit description of the Langlands parameters of such representations.

We examine an extreme class of irreducible cuspidal representations of the general linear group $G = \mathrm{GL}_n(F)$ over a non-Archimedean local field F , those designated “epipelagic” by Reeder and Yu [26]. A particular arithmetic interest of such representations was first identified by Gross and Reeder in [18], who dubbed them “simple cuspidals”. The authors of [18] asked us whether anything could be said about the Langlands parameters. This paper is the response.

The context of [26] is quite general but, for the group $\mathrm{GL}_n(F)$, an exercise in techniques going back to [1] shows that a cuspidal representation π is epipelagic if and only if the exponential Swan conductor $\mathrm{Sw}(\pi)$ is equal to 1. We start from that point.

1. If $n \geq 1$ is an integer, let $\mathcal{A}_n(F)$ be the set of equivalence classes of irreducible, cuspidal (complex) representations of $G = \mathrm{GL}_n(F)$. Let $\mathcal{G}_n(F)$ be the set of equivalence classes of irreducible, smooth, n -dimensional representations of the Weil group \mathcal{W}_F of some separable algebraic closure \bar{F}/F of F . We denote the Langlands correspondence $\mathcal{A}_n(F) \rightarrow \mathcal{G}_n(F)$ by $\pi \mapsto {}^L\pi$. Let ${}^1\mathcal{A}_n(F)$ be the set of $\pi \in \mathcal{A}_n(F)$ with $\mathrm{Sw}(\pi) = 1$ and define ${}^1\mathcal{G}_n(F)$ in the same way. The Langlands correspondence preserves Swan conductors and so maps ${}^1\mathcal{A}_n(F)$ bijectively to ${}^1\mathcal{G}_n(F)$.

It is easy to describe the elements of ${}^1\mathcal{A}_n(F)$ in terms of the standard model of [14]. A representation $\pi \in \mathcal{A}_n(F)$, not of level zero, contains a simple character θ . If we choose a character ψ_F of F satisfying $c(\psi_F) = -1$ (see §1 for this notation), then θ is attached to a simple stratum $[\mathfrak{a}, l, 0, \alpha]$ in the matrix algebra $A = M_n(F)$. The condition $\mathrm{Sw}(\pi) = 1$ translates into l being 1, the hereditary order \mathfrak{a} being minimal and the field extension $F[\alpha]/F$ being totally ramified of degree n . Fixing such a stratum $[\mathfrak{a}, 1, 0, \alpha]$, one may list the elements π of ${}^1\mathcal{A}_n(F)$ to which it is attached. This yields a classification of the elements of ${}^1\mathcal{A}_n(F)$ in terms of properties of local constants. Passing across the Langlands correspondence, one gets a parallel classification of the elements of ${}^1\mathcal{G}_n(F)$.

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2. On the Galois side, that classification exhibits a lamentable opacity. We set ourselves the task of describing explicitly the representation $\sigma = {}^L\pi$, $\pi \in {}^1\mathcal{A}_n(F)$, in terms of the structure of π .

There are not many cases where such descriptions have been seriously attempted. The essentially tame case is exhaustively worked out in [7], [8], [10]. The wildly ramified case has only been considered in prime dimension [9], [20], Kutzko [24], Mœglin [25], with the aim of verifying the Langlands conjecture, as it was at the time. The dissection of the representations was carried no further than necessary for that purpose. A thorough analysis, even of a special case, seems overdue. The epipelagic representations exhibit sufficient complexity to make the exercise worthwhile and can be used to generate further families of examples.

3. Let $\sigma \in {}^1\mathcal{G}_n(F)$ and $\pi \in {}^1\mathcal{A}_n(F)$ satisfy ${}^L\pi = \sigma$. Let p be the residual characteristic of F , and write $n = ep^r$, for integers e, r with e not divisible by p . There is then a totally ramified extension K/F of degree e , and a representation $\tau \in {}^1\mathcal{G}_{p^r}(F)$, such that τ induces σ . The pair $(K/F, \tau)$ is uniquely determined up to conjugation in \mathcal{W}_F . The theory of tame lifting enables us to specify K in terms of π , and also the representation $\rho \in {}^1\mathcal{A}_{p^r}(K)$ such that $\tau = {}^L\rho$.

4. We therefore concentrate on the case ${}^L\pi = \sigma \in {}^1\mathcal{G}_{p^r}(F)$. Here, a conductor estimate shows that σ is *primitive*, and so covered by Koch's seminal work [23].

Let $\bar{\sigma}$ be the *projective* representation of \mathcal{W}_F determined by σ , and define a finite Galois extension E/F by $\mathcal{W}_E = \text{Ker } \bar{\sigma}$. The restriction of σ to \mathcal{W}_E is therefore a multiple of a character ξ_σ of \mathcal{W}_E . Let T/F be the maximal tamely ramified sub-extension of E/F . Following [23], the restriction σ_T of σ to \mathcal{W}_T is irreducible. The Galois group of E/T is elementary abelian of order p^{2r} , and σ_T is the unique irreducible representation of \mathcal{W}_T containing ξ_σ . The classification theory of [23] then rests on the fact that $\text{Gal}(E/T)$ provides a symplectic representation of $\text{Gal}(T/F)$ over the field \mathbb{F}_p of p elements.

There is a helpful element of structure here. If K/F is a finite tame extension, let σ_K denote the (irreducible) restriction of σ to \mathcal{W}_K . Let $D(\sigma_K)$ be the group of characters χ of \mathcal{W}_K for which $\chi \otimes \sigma_K \cong \sigma_K$. One then has $|D(\sigma_K)| \leq p^{2r}$, with equality if and only if K contains T . Moreover, $D(\sigma_T)$ is the dual of the Galois group $\text{Gal}(E/T) = \mathcal{W}_T/\mathcal{W}_E$.

We can make no further progress working only with Galois representations and have to switch to the other side. If K/F is a finite tame extension, we may define $\pi_K \in \mathcal{A}_{p^r}(K)$ by the relation ${}^L\pi_K = \sigma_K$. Using results from [4], [5] on tame lifting, we can describe π_K explicitly, in terms of π . We have to find the least tame extension T/F for which there are p^{2r} characters ϕ of T^\times satisfying $\phi\pi_T \cong \pi_T$. An elaborate calculation is required. The outcome is an explicit polynomial, with splitting field T . The roots of the polynomial define the non-trivial elements of $D(\sigma_T)$, as characters of T^\times . Via class field theory, they determine the abelian extension E/T .

The structure of the argument is worthy of comment. The initial steps, from the Galois theory of local fields, are fairly straightforward. We only use the early parts of [23], not going much beyond the group-theoretic arguments of Rigby [27]. The main calculations, leading to the polynomial determining the field T , are quite involved and use a lot of machinery from [14], but they are essentially elementary in nature. The depth lies in the facility with which we can move from side to side, via the Langlands correspondence, without loss of explicit detail. This is absolutely reliant on a substantial portion of the theory of tame lifting, as developed in [4], [6] and [12].

5. Having identified the fields T/F and E/T , it remains to understand the “ p -central character” ξ_σ . Starting from the description of T and $D(\sigma_T)$, classical methods of local field theory and a local constant calculation yield an expression for ξ_σ on the unit group U_E^1 . We shall see that the datum $(\xi_\sigma|_{U_E^1}, \det \sigma)$ determines σ up to tensoring with an unramified character of order dividing p^r (6.2 below). Since $\text{Sw}(\sigma)$ is not divisible by p , this uncertainty is removed by a simple local constant relation.

It is possible to specify ξ_σ completely by calculating one further special value, but the details are voluminous. Nothing much is gained, so we have omitted them.

1. PRELIMINARIES AND NOTATION

We set down our standard notation and recall some basic facts.

1.1. Throughout, F is a non-Archimedean local field with discrete valuation ring \mathfrak{o}_F . The maximal ideal of \mathfrak{o}_F is \mathfrak{p}_F , the residue field is $\mathbb{k}_F = \mathfrak{o}_F/\mathfrak{p}_F$, $q = |\mathbb{k}_F|$, and $v_F : F^\times \rightarrow \mathbb{Z}$ is the normalized additive valuation. The characteristic of \mathbb{k}_F is p . The unit group is $U_F = \mathfrak{o}_F^\times$ and $U_F^k = 1 + \mathfrak{p}_F^k$, $k \geq 1$. Also, μ_F is the group of roots of unity in F , of order relatively prime to p .

If \mathfrak{a} is a hereditary \mathfrak{o}_F -order in some matrix algebra $M_n(F)$ and $\mathfrak{p}_\mathfrak{a}$ is the Jacobson radical of \mathfrak{a} , then $U_\mathfrak{a} = \mathfrak{a}^\times$ and $U_\mathfrak{a}^k = 1 + \mathfrak{p}_\mathfrak{a}^k$, $k \geq 1$.

Let \bar{F}/F be a separable algebraic closure of F . Let \mathcal{W}_F be the Weil group of \bar{F}/F and \mathcal{P}_F the wild inertia subgroup of \mathcal{W}_F . We identify the group of smooth characters of \mathcal{W}_F with that of F^\times via class field theory, switching between the two viewpoints as convenient.

Let $\hat{\mathcal{W}}_F$ be the set of equivalence classes of irreducible smooth representations of \mathcal{W}_F . For an integer $n \geq 1$, let $\mathcal{G}_n(F)$ be the set of $\sigma \in \hat{\mathcal{W}}_F$ of dimension n . Let $\mathcal{A}_n(F)$ be the set of equivalence classes of irreducible *cuspidal* representations of $\text{GL}_n(F)$. The Langlands correspondence gives a bijection $\mathcal{A}_n(F) \rightarrow \mathcal{G}_n(F)$ which we denote by $\pi \mapsto {}^L\pi$.

1.2. Let ψ be a smooth character of F , $\psi \neq 1$. Define $c(\psi)$ to be the greatest integer k such that $\mathfrak{p}_F^{-k} \subset \text{Ker } \psi$. We shall mostly work with the case $c(\psi) = -1$, where ψ is trivial on \mathfrak{p}_F but not on \mathfrak{o}_F .

Let $n \geq 1$ be an integer, let $\pi \in \mathcal{A}_n(F)$. The *Godement-Jacquet local constant* $\varepsilon(\pi, s, \psi)$ of π , relative to the character ψ and a complex variable s , takes the form (Godement-Jacquet [17], Jacquet [21])

$$\varepsilon(\pi, s, \psi) = q^{-sf(\pi, \psi)} \varepsilon(\pi, 0, \psi),$$

for an integer $f(\pi, \psi)$ such that

$$f(\pi, \psi) = a(\pi) + nc(\psi),$$

where the *Artin conductor* $a(\pi)$ is independent of ψ . Set

$$\text{sw}(\pi) = a(\pi) - n.$$

Thus $\text{sw}(\pi) = f(\pi, \psi)$ when $c(\psi) = -1$. The integer $\text{sw}(\pi)$ is the same as the Swan conductor $\text{Sw}(\pi)$ *except* when $n = 1$ and π is an unramified character of F^\times . In that case, $\text{sw}(\pi) = -1$ and $\text{Sw}(\pi) = 0$.

We denote by ${}^1\mathcal{A}_n(F)$ the set of $\pi \in \mathcal{A}_n(F)$ for which $\text{sw}(\pi) = 1$.

1.3. Let $\sigma \in \mathcal{G}_n(F)$. The *Langlands-Deligne local constant* $\varepsilon(\sigma, s, \psi)$ takes the form [9], Tate [30],

$$\varepsilon(\sigma, s, \psi) = q^{-sf(\sigma, \psi)} \varepsilon(\sigma, 0, \psi),$$

with $f(\sigma, \psi) = a(\sigma) + nc(\psi)$, the Artin conductor $a(\sigma)$ being independent of ψ . Likewise set

$$\text{sw}(\sigma) = a(\sigma) - n, \quad n = \dim \sigma.$$

This relates to the Swan conductor $\text{Sw}(\sigma)$ as before. Let ${}^1\mathcal{G}_n(F)$ be the set of $\sigma \in \mathcal{G}_n(F)$ with $\text{sw}(\sigma) = 1$. The Langlands correspondence satisfies $\text{sw}({}^L\pi) = \text{sw}(\pi)$, $\pi \in \mathcal{A}_n(F)$, so it maps ${}^1\mathcal{A}_n(F)$ bijectively to ${}^1\mathcal{G}_n(F)$.

We make frequent use of the following fact.

Induction Formula. *Let E/F be a finite separable extension, set $e = e(E|F)$, $f = f(E|F)$, and let $\mathfrak{D}_{E/F} = \mathfrak{p}_E^d$ be the relative different of E/F . If $\tau \in \widehat{W}_E$ has dimension m and $\sigma = \text{Ind}_{E/F} \tau$, then*

$$(1.3.1) \quad \text{sw}(\sigma) = (\text{sw}(\tau) + m(1-e+d))f.$$

Proof. If ψ is a smooth character of F , $\psi \neq 1$, and $\psi_E = \psi \circ \text{Tr}_{E/F}$, then

$$(1.3.2) \quad c(\psi_E) = d + ec(\psi).$$

According to [9] 30.4 Corollary, the quotient $\varepsilon(\sigma, s, \psi)/\varepsilon(\tau, s, \psi_E)$ is independent of s . The result follows from a simple computation. \square

2. CLASSIFICATION

Let $n \geq 1$ be an integer. In this section, we classify the elements of ${}^1\mathcal{A}_n(F)$ using the method of [14]. We translate this into a classification in terms of properties of local constants. Via the Langlands correspondence, this gives a parallel classification of the elements of ${}^1\mathcal{G}_n(F)$.

Since we work in the context of [14], we need to choose a smooth character ψ_F of F with $c(\psi_F) = -1$. Thus, in the language of [14], the character ψ_F “has level one”. Throughout, we write $G = \text{GL}_n(F)$ and $A = \text{M}_n(F)$.

2.1. Let $\pi \in \mathcal{A}_n(F)$. Thus π contains a *simple character* θ in G , in the sense of [14]. This simple character is uniquely determined up to G -conjugation. It is trivial if and only if $\text{sw}(\pi) = 0$. Assuming θ to be non-trivial, it is attached via ψ_F to a simple stratum $[\mathfrak{a}, l, 0, \beta]$ in A : in the notation of [14], $\theta \in \mathcal{C}(\mathfrak{a}, \beta, \psi_F)$. The stratum $[\mathfrak{a}, l, 0, \beta]$ is *m-simple*, in that \mathfrak{a} is maximal among hereditary \mathfrak{o}_F -orders in A , stable under conjugation by $F[\beta]^\times$. Correspondingly, one says that θ is m-simple.

If $e_{\mathfrak{a}}$ denotes the F -period of the hereditary \mathfrak{o}_F -order \mathfrak{a} , then

$$\text{sw}(\pi) = ln/e_{\mathfrak{a}} \geq 1,$$

[5] 6.1 Lemma 2. Since the stratum $[\mathfrak{a}, l, 0, \beta]$ is m-simple, the period $e_{\mathfrak{a}}$ equals the ramification index $e(F[\beta]|F)$ of the field extension $F[\beta]/F$. We encapsulate these remarks:

Lemma 1. *Let $\pi \in \mathcal{A}_n(F)$. The following conditions are equivalent:*

- (1) $\text{sw}(\pi) = 1$;
- (2) π contains an m -simple character $\theta \in \mathcal{C}(\mathfrak{a}, \alpha, \psi_F)$, where $[\mathfrak{a}, 1, 0, \alpha]$ is an m -simple stratum in A such that $e_{\mathfrak{a}} = n$.

When these conditions are satisfied, the field extension $F[\alpha]/F$ is totally ramified of degree n .

We write down the various groups and the simple characters attached to the simple stratum $[\mathfrak{a}, 1, 0, \alpha]$ in the machinery of [14] Chapter 3. Let $\text{tr}_A : A \rightarrow F$ be the matrix trace and write $\psi_A = \psi_F \circ \text{tr}_A$.

Lemma 2. *Let $[\mathfrak{a}, 1, 0, \alpha]$ be an m -simple stratum in A with $e_{\mathfrak{a}} = n$.*

- (1) *The associated groups are*

$$H^1(\alpha, \mathfrak{a}) = J^1(\alpha, \mathfrak{a}) = U_{\mathfrak{a}}^1.$$

The set $\mathcal{C}(\mathfrak{a}, \alpha, \psi_F)$ of simple characters has only the one element

$$\theta_{\alpha} : 1+x \mapsto \psi_A(\alpha x), \quad 1+x \in U_{\mathfrak{a}}^1.$$

The G -normalizer of the character θ_{α} is $\mathbf{J}_{\alpha} = F[\alpha]^{\times} U_{\mathfrak{a}}^1$.

- (2) *Let Ψ be a character of \mathbf{J}_{α} such that $\Psi|_{U_{\mathfrak{a}}^1} = \theta_{\alpha}$. The G -representation*

$$\pi_{\Psi} = c\text{-Ind}_{\mathbf{J}_{\alpha}}^G \Psi$$

is irreducible and cuspidal. The map $\Psi \mapsto \pi_{\Psi}$ is a bijection between the set of characters of \mathbf{J}_{α} extending θ_{α} and the set of $\pi \in \mathcal{A}_n(F)$ containing θ_{α} .

Proof. Part (1) follows on working through the definitions in Chapter 3 of [14]. Part (2) is an instance of [14] (8.4.1) or [12] 4.1. \square

In part (1) of the lemma, the character θ_{α} determines only the coset $\alpha + \mathfrak{a} = \alpha U_{\mathfrak{a}}^1$. In the situation of part (2), it will be easier to say that π contains the m -simple stratum $[\mathfrak{a}, 1, 0, \alpha]$ (relative to the character ψ_F), rather than that it contains θ_{α} .

Proposition.

- (1) *If $\pi \in {}^1\mathcal{A}_n(F)$, $n > 1$, the central character ω_{π} of π is tamely ramified.*
- (2) *Let ω be a tamely ramified character of F^{\times} and $[\mathfrak{a}, 1, 0, \alpha]$ an m -simple stratum in A , with $e_{\mathfrak{a}} = n$. There exist exactly n representations $\pi \in {}^1\mathcal{A}_n(F)$ containing $[\mathfrak{a}, 1, 0, \alpha]$ and having central character $\omega_{\pi} = \omega$.*

Proof. Let $\pi \in {}^1\mathcal{A}_n(F)$ contain the m -simple stratum $[\mathfrak{a}, 1, 0, \alpha]$. We have $U_F^1 = F^{\times} \cap U_{\mathfrak{a}}^n \subset \text{Ker } \theta_{\alpha}$. Therefore ω_{π} is trivial on U_F^1 , as required for (1). The group \mathbf{J}_{α} contains $F^{\times} U_{\mathfrak{a}}^1$ with index n , so (2) follows from Lemma 2. \square

2.2. If $\pi \in \mathcal{A}_n(F)$ and if χ is a character of F^{\times} , then $\chi\pi$ will denote the representation $g \mapsto \chi(\det g)\pi(g)$. In particular, $\chi\pi \in \mathcal{A}_n(F)$.

We compute Godement-Jacquet local constants. The method originates in [3], [1], but the summary in [5] 6.1 is closer to our current notation.

Lemma. *If $\pi = c\text{-Ind}_{\mathbf{J}_\alpha}^G \Psi_\pi$, as in 2.1 Lemma 2, then*

- (1) $\varepsilon(\pi, \frac{1}{2}, \psi_F) = \Psi_\pi(\alpha)^{-1} \psi_A(\alpha)$ and,
- (2) *if χ is a tamely ramified character of F^\times , then*

$$\begin{aligned} \text{sw}(\chi\pi) &= \text{sw}(\pi) = 1, \\ \varepsilon(\chi\pi, s, \psi_F) &= \chi(\det \alpha)^{-1} \varepsilon(\pi, s, \psi_F). \end{aligned}$$

We may now strengthen the description of 2.1.

Proposition. *For $i = 1, 2$, let $\pi_i \in {}^1\mathcal{A}_n(F)$, and let ω_i be the central character of π_i . Let π_i contain the m -simple stratum $[\mathfrak{a}, 1, 0, \alpha_i]$. The representations π_1, π_2 are equivalent if and only if the following conditions are satisfied:*

- (a) $\det \alpha_1 \equiv \det \alpha_2 \pmod{U_F^1}$,
- (b) $\omega_1 = \omega_2$, and
- (c) $\varepsilon(\pi_1, \frac{1}{2}, \psi_F) = \varepsilon(\pi_2, \frac{1}{2}, \psi_F)$.

Proof. Let $\mathfrak{p}_\mathfrak{a}$ be the Jacobson radical of \mathfrak{a} . If the π_i are equivalent, the m -simple characters θ_{α_i} are conjugate in G or, equivalently, the cosets $\alpha_i U_\mathfrak{a}^1$ are $U_\mathfrak{a}$ -conjugate. To elucidate this condition, we use a matrix normal form calculation. We may assume that \mathfrak{a} is the ring of matrices, with entries in \mathfrak{o}_F , which become upper triangular when reduced modulo \mathfrak{p}_F . The ideal $\mathfrak{p}_\mathfrak{a}$ then consists of the matrices which are strictly upper triangular modulo \mathfrak{p}_F . We have $\alpha_i^{-1} \mathfrak{a} = \mathfrak{p}_\mathfrak{a}$ for both values of i , this being part of the definition of a simple stratum. A matrix manipulation shows that α_i^{-1} is $U_\mathfrak{a}$ -conjugate to a matrix $a_i u_i$, where

- (1) $u_i \in U_\mathfrak{a}^1$;
- (2) $(a_i)_{j,j+1} = 1$, for $1 \leq j < n$;
- (3) $(a_i)_{n,1}$ is a prime element ϖ_i of F ;
- (4) all other entries of a_i are zero.

The cosets $a_i U_\mathfrak{a}^1$ are then $U_\mathfrak{a}$ -conjugate if and only if $\varpi_1 \equiv \varpi_2 \pmod{U_F^1}$, and this is equivalent to condition (a) of the proposition.

We are thus reduced to the case $\alpha_1 = \alpha_2 = \alpha$, say. Set $E = F[\alpha]$. There are characters Ψ_i of $\mathbf{J}_\alpha = E^\times U_\mathfrak{a}^1$ such that $\Psi_i|_{U_\mathfrak{a}^1} = \theta_\alpha$ and $\pi_i \cong c\text{-Ind}_{\mathbf{J}_\alpha}^G \Psi_i$, $i = 1, 2$. Moreover, $\pi_1 \cong \pi_2$ if and only if $\Psi_1 = \Psi_2$. Condition (b) of the proposition is equivalent to $\Psi_1|_{F^\times} = \Psi_2|_{F^\times}$ (2.1 Lemma 2). Using the last lemma, condition (c) of the proposition is now equivalent to $\Psi_1(\alpha) = \Psi_2(\alpha)$. Since α generates the finite cyclic group $\mathbf{J}_\alpha / F^\times U_\mathfrak{a}^1$, the result follows. \square

Remark. Let ω be a tamely ramified character of F^\times . Taking account of 2.1 Proposition, we see that the set ${}^1\mathcal{A}_n(F)$ has exactly $n(q-1)$ elements π such that $\omega_\pi = \omega$.

2.3. We use the Langlands correspondence to translate the classification from 2.2 Proposition in terms of representations of \mathcal{W}_F .

Lemma. *Let $\sigma \in \widehat{\mathcal{W}}_F$ and suppose that $\text{sw}(\sigma) \geq 1$. There exists $\gamma_\sigma \in F^\times$ such that*

$$\varepsilon(\chi \otimes \sigma, s, \psi_F) = \chi(\gamma_\sigma)^{-1} \varepsilon(\sigma, s, \psi_F),$$

for any tamely ramified character χ of \mathcal{W}_F . This property determines the coset $\gamma_\sigma U_F^1$ uniquely.

Proof. The lemma is a case of the main result of [16]. \square

Proposition. *Let $\sigma \in {}^1\mathcal{G}_n(F)$. Define $\pi \in {}^1\mathcal{A}_n(F)$ by ${}^L\pi = \sigma$. If $[\mathfrak{a}, 1, 0, \alpha]$ is an m -simple stratum contained in π , then*

$$\begin{aligned}\gamma_\sigma &\equiv \det \alpha \pmod{U_F^1}, \\ \det \sigma &= \omega_\pi, \\ \varepsilon(\sigma, s, \psi_F) &= \varepsilon(\pi, s, \psi_F).\end{aligned}$$

Moreover, π is the only element of $\mathcal{A}_n(F)$ with these properties.

Proof. The Langlands correspondence $\mathcal{A}_n(F) \rightarrow \mathcal{G}_n(F)$ takes the central character to the determinant, it preserves the local constant and respects twisting with characters. The result therefore follows from the lemma and 2.2 Lemma. \square

Combining the proposition with 2.2 Proposition, we get:

Corollary. *Let $\sigma_1, \sigma_2 \in {}^1\mathcal{G}_n(F)$. The representations σ_1, σ_2 are equivalent if and only if the following conditions hold:*

- (1) $\gamma_{\sigma_1} \equiv \gamma_{\sigma_2} \pmod{U_F^1}$,
- (2) $\det \sigma_1 = \det \sigma_2$, and
- (3) $\varepsilon(\sigma_1, s, \psi_F) = \varepsilon(\sigma_2, s, \psi_F)$.

Reflection. It is not difficult to reconstruct $\pi \in {}^1\mathcal{A}_n(F)$ from properties of local constants. In particular, the function $\chi \mapsto \varepsilon(\chi\pi, \frac{1}{2}, \psi)$, with χ ranging over the tame characters of F^\times , reveals completely the simple character contained in π . For $\sigma \in {}^1\mathcal{G}_n(F)$, the corresponding property gives the coset $\gamma_\sigma U_F^1$ and that determines the simple character in π , ${}^L\pi = \sigma$. According to the Ramification Theorem [6] 8.2 or [12] 6.1, it must also determine the restriction of σ to the wild inertia subgroup of \mathcal{W}_F . The process by which it does this is the central concern of the paper.

3. PRIMITIVITY

While the results of §2 classify the elements of ${}^1\mathcal{G}_n(F)$, they do not describe them at all effectively. We investigate further.

3.1. Any $\sigma \in {}^1\mathcal{G}_n(F)$ is *totally ramified*, in the following sense.

Lemma. *Let $\sigma \in {}^1\mathcal{G}_n(F)$ and let χ be an unramified character of \mathcal{W}_F . If the representations $\chi \otimes \sigma, \sigma$ are equivalent then $\chi = 1$.*

Proof. By 2.3 Proposition, $v_F(\gamma_\sigma) = -1$ in this case. If the unramified character χ is non-trivial, then $\varepsilon(\chi \otimes \sigma, s, \psi_F) = \chi(\gamma_\sigma)^{-1} \varepsilon(\sigma, s, \psi_F) \neq \varepsilon(\sigma, s, \psi_F)$, whence $\chi \otimes \sigma \not\cong \sigma$. \square

Equivalently, a representation $\sigma \in {}^1\mathcal{G}_n(F)$ restricts irreducibly to the inertia subgroup of \mathcal{W}_F .

Proposition 1. *Let $n = ep^r$, for integers e, r such that p does not divide e . Let $\sigma \in {}^1\mathcal{G}_n(F)$.*

- (1) *There exists a totally ramified extension K/F , of degree e , and a representation $\tau \in \widehat{\mathcal{W}}_K$ such that $\sigma \cong \text{Ind}_{K/F} \tau$. This relation determines the pair $(K/F, \tau)$ uniquely up to \mathcal{W}_F -conjugation.*
- (2) *The representation τ satisfies $\text{sw}(\tau) = 1$.*

Proof. Part (1) is an instance of [6] 8.6 Proposition. Part (2) follows directly from the Induction Formula (1.3.1). \square

We describe the representation τ in the manner of §2. Let $\sigma = {}^L\pi$, $\pi \in {}^1\mathcal{A}_n(F)$. As in 2.1, π contains the simple character $\theta_\alpha \in \mathcal{C}(\mathfrak{a}, \alpha, \psi_F)$, for a simple stratum $[\mathfrak{a}, 1, 0, \alpha]$ in $A = M_n(F)$.

Let T/F be the maximal tamely ramified sub-extension of $F[\alpha]/F$. Let $B \cong M_{p^r}(T)$ be the A -centralizer of T and set $\mathfrak{b} = \mathfrak{a} \cap B$. Write $\psi_T = \psi_F \circ \text{Tr}_{T/F}$. The quadruple $[\mathfrak{b}, 1, 0, \alpha]$ is then an m -simple stratum in B and the set $\mathcal{C}(\mathfrak{b}, \alpha, \psi_T)$ has only one element θ_α^T , as in 2.1 Lemma 2.

We take $(K/F, \tau)$ as in Proposition 1.

Proposition 2.

- (1) *The fields T, K are F -isomorphic.*
- (2) *Let $\rho \in {}^1\mathcal{A}_{p^r}(K)$ satisfy ${}^L\rho = \tau$. There is an F -isomorphism $f : K \rightarrow T$ such that the representation $f_*\rho \in {}^1\mathcal{A}_{p^r}(T)$ contains the simple character θ_α^T .*

Proof. Part (1) follows from the Tame Parameter Theorem of [12] 6.3 and the second from [12] 6.2. \square

To save notation, we identify T with K via the isomorphism f , and let $\det_B : B^\times \rightarrow K^\times$ be the determinant map.

Corollary. *Let 1_K be the trivial character of \mathcal{W}_K . Set $R_{K/F} = \text{Ind}_{K/F} 1_K$ and $\delta_{K/F} = \det R_{K/F}$. The representation τ satisfies:*

- (1) $\gamma_\tau \equiv \det_B \alpha \pmod{U_K^1}$;
- (2) $\det \tau|_{F^\times} = \delta_{K/F}^{-p^r} \det \sigma$;
- (3) $\varepsilon(\sigma, s, \psi_F)/\varepsilon(\tau, s, \psi_K) = (\varepsilon(R_{K/F}, s, \psi_F)/\varepsilon(1_K, s, \psi_K))^{p^r}$.

These relations determine τ uniquely.

Proof. Part (1) is implied by Proposition 2 and 2.3 Proposition. For parts (2) and (3), see for instance [9] 29.2, (29.4.1). Uniqueness follows from 2.3 Corollary. \square

The values of the Langlands constant $\varepsilon(R_{K/F}, s, \psi_F)/\varepsilon(1_K, s, \psi_K)$ are tabulated in, for example, [2] 10.1. The corollary allows one to specify the representation ρ , such that ${}^L\rho = \tau$, in the manner of 2.2.

3.2. The results of 3.1 reduce us to the case where $\dim \sigma$ is a power of p . In particular, σ is *totally wildly ramified*, in that it remains irreducible on restriction to the wild inertia subgroup \mathcal{P}_F of \mathcal{W}_F .

Recall that a representation $\tau \in \widehat{\mathcal{W}}_F$ is *imprimitive* if there exists a finite separable extension K/F , $K \neq F$, and a representation $\rho \in \widehat{\mathcal{W}}_K$ such that $\tau \cong \text{Ind}_{K/F} \rho$. Thus τ is called *primitive* if it is not imprimitive.

Proposition. *If $\sigma \in {}^1\mathcal{G}_{p^r}(F)$, for an integer $r \geq 1$, then σ is primitive.*

Proof. Suppose, for a contradiction, that $\sigma \cong \text{Ind}_{K/F} \tau$, for a finite extension K/F , $K \neq F$, and $\tau \in \widehat{\mathcal{W}}_K$. Thus $[K:F] = p^a$ and $\dim \tau = p^b$, for integers a, b such that $a+b = r$. By (1.3.1),

$\text{sw}(\sigma) = 1$ is divisible by $f(K|F)$, so K/F must be totally ramified. Since $[K:F]$ is a power of p , the extension K/F is totally wildly ramified. If $\mathfrak{D}_{K/F} = \mathfrak{p}_K^{d(K|F)}$ is the different of K/F , this means $d(K|F) \geq [K:F] = p^a$. Therefore

$$1 = \text{sw}(\sigma) \geq \text{sw}(\tau) + p^b(1 - p^a + p^a),$$

or

$$\text{sw}(\tau) \leq 1 - p^b \leq 0.$$

The only possibility here is $p^b = \dim \tau = 1$ and τ tamely ramified. The character τ of K^\times is then of the form $\xi \circ \text{N}_{K/F}$, for a tamely ramified character ξ of F^\times . The character ξ must occur as a component of $\text{Ind}_{K/F} \tau$, which is therefore reducible. This contradicts our hypothesis. \square

4. STRUCTURE OF PRIMITIVE REPRESENTATIONS

Following 3.2 Proposition, a representation $\sigma \in {}^1\mathcal{G}_{p^r}(F)$, $r \geq 1$, is necessarily primitive. Primitive representations are described in [23]. We now see how σ fits into the scheme of [23], as a basis for the detailed analysis of the next section.

4.1. We make some remarks related to Clifford theory. For $\tau \in \widehat{\mathcal{W}}_F$, let $D(\tau)$ denote the group of characters χ of \mathcal{W}_F such that $\chi \otimes \tau \cong \tau$.

Lemma. *Let $\tau \in \widehat{\mathcal{W}}_F$ have dimension n . The abelian group $D(\tau)$ has exponent dividing n and order at most n^2 .*

Proof. Let $\chi \in D(\tau)$. The relation $\chi \otimes \sigma \cong \sigma$ implies $\det \sigma = \det(\chi \otimes \sigma) = \chi^n \det \sigma$, whence $\chi^n = 1$. Let $\tilde{\tau}$ denote the contragredient of τ , and let χ be a character of \mathcal{W}_F . Frobenius Reciprocity yields

$$\text{Hom}_{\mathcal{W}_F}(\chi \otimes \tau, \tau) \cong \text{Hom}_{\mathcal{W}_F}(\chi, \tilde{\tau} \otimes \tau).$$

We deduce that $\chi \in D(\tau)$ if and only if χ is a component of $\tilde{\tau} \otimes \tau$. This proves the second assertion. \square

Proposition. *Let $\tau \in \widehat{\mathcal{W}}_F$ be totally wildly ramified.*

- (1) *If $\chi \in D(\tau)$ is tamely ramified, then $\chi = 1$.*
- (2) *If K/F is a finite tame extension, the canonical map*

$$\begin{aligned} D(\tau) &\longrightarrow D(\tau_K), \\ \chi &\longmapsto \chi_K = \chi \circ \text{N}_{K/F}, \end{aligned}$$

is injective. If K/F is Galois, the image of $D(\tau)$ is the set of $\text{Gal}(K/F)$ -fixed points in $D(\tau_K)$.

Proof. Let $\chi \in D(\tau)$ be tamely ramified, $\chi \neq 1$. The kernel of χ is then \mathcal{W}_K , for a cyclic tame extension K/F and, by Clifford theory, $\tau \cong \text{Ind}_{K/F} \rho$, for some $\rho \in \widehat{\mathcal{W}}_K$. The restriction of τ to \mathcal{W}_K is therefore not irreducible. However, \mathcal{W}_K contains $\mathcal{P}_K = \mathcal{P}_F$ so this contradicts our hypothesis. Part (1) is proved.

In part (2), let $\chi \in D(\tau)$. View χ as a character of F^\times and suppose that $\chi_K = 1$. The field norm $\text{N}_{K/F}$ induces a surjection $U_K^1 \rightarrow U_F^1$, so χ is trivial on U_F^1 . Part (1) implies $\chi = 1$. For

the second assertion, write $\Gamma = \text{Gal}(K/F)$. By transitivity and the first assertion, we need only treat the case where Γ is cyclic. Let $\phi \in D(\tau_K)^\Gamma$. This implies $\phi = \chi_K$, for a character χ of F^\times . We then have $(\chi \otimes \tau)_K \cong \tau_K$, whence it follows that $\chi \otimes \tau \cong \delta \otimes \tau$, for a character δ of F^\times such that $\delta_K = 1$. In particular, $\delta^{-1}\chi \in D(\tau)$ while $(\delta^{-1}\chi)_K$ agrees with ϕ on U_K^1 . Therefore $(\delta^{-1}\chi)_K = \phi$, as required. \square

4.2. We recall some facts about representations of *Heisenberg type*. Let $\tau \in \widehat{\mathcal{W}}_F$ be totally ramified of dimension p^r , for some $r \geq 1$. Let $\bar{\tau} : \mathcal{W}_F \rightarrow \text{PGL}_{p^r}(\mathbb{C})$ be the associated projective representation. Thus $\text{Ker } \bar{\tau} = \mathcal{W}_E$, for a finite Galois extension E/F , and the restriction of τ to \mathcal{W}_E is a multiple of a character ξ_τ of \mathcal{W}_E . We call E the *p-kernel field* of τ and ξ_τ the *p-central character* of τ .

We are concerned with the case where $\Delta = \text{Gal}(E/F) = \text{Im } \bar{\tau} \cong (\mathbb{Z}/p\mathbb{Z})^{2r}$. The quantity $\xi_\tau[x, y] = \xi_\tau(xyx^{-1}y^{-1})$, $x, y \in \mathcal{W}_F$, is then of order dividing p . We identify the group $\mu_p(\mathbb{C})$ of p -th roots of unity in \mathbb{C} with the additive group of the field \mathbb{F}_p of p elements. The group Δ is a vector space over \mathbb{F}_p and the identification $\mu_p(\mathbb{C}) = \mathbb{F}_p$ allows us to view the pairing $h_\tau : (x, y) \mapsto \xi_\tau[x, y]$ as a bilinear form $\Delta \times \Delta \rightarrow \mathbb{F}_p$. This form h_τ is alternating and nondegenerate. One says that τ is of *Heisenberg type*, since the image $\tau(\mathcal{W}_F)$ is a Heisenberg group.

Lemma. *Let $\tau \in \widehat{\mathcal{W}}_F$ be of Heisenberg type, with p-kernel field E , p-central character ξ_τ and dimension p^r .*

- (1) *If θ is an irreducible representation of \mathcal{W}_F such that $\theta|_{\mathcal{W}_E}$ contains ξ_τ , then $\theta \cong \tau$.*
- (2) *Let Δ' be a subgroup of Δ , of order p^r , such that $h_\tau[x, y] = 0$, for all $x, y \in \Delta'$. Let $\mathcal{W}_{E'}$ be the inverse image of Δ' in \mathcal{W}_F .*
 - (a) *There exists a character ψ of $\mathcal{W}_{E'}$ such that $\psi|_{\mathcal{W}_E} = \xi_\tau$.*
 - (b) *For any such character ψ , the induced representation $\text{Ind}_{E'/F} \psi$ is equivalent to τ . In particular, $\text{Ind}_{E/F} \xi_\tau$ is a sum of p^r copies of τ .*
- (3) *The group $D(\tau)$ has order p^{2r} and consists of all characters of F^\times trivial on norms from E . That is, $D(\tau)$ is the group $\widehat{\Delta}$ of characters of Δ .*

Proof. This is an exercise, for which hints may be found in, for example, [2] 8.3. \square

We recall the structure of the irreducible primitive representations of \mathcal{W}_F . Leaving aside the trivial one-dimensional case, any such representation has dimension p^r , $r \geq 1$, and is totally wildly ramified.

Proposition. *Let $\sigma \in \widehat{\mathcal{W}}_F$ be primitive of dimension p^r , $r \geq 1$. Let E/F be the p-kernel field of σ , and let T/F be the maximal tamely ramified sub-extension of E/F . Let $\Pi = \text{Gal}(E/T)$ and $\Gamma = \text{Gal}(T/F)$. Let ξ_σ be the p-central character of σ .*

- (1) *The restriction σ_T of σ to \mathcal{W}_T is irreducible of Heisenberg type.*
- (2) *The alternating form h_σ is invariant under Γ ,*

$$h_\sigma(\gamma u, \gamma v) = h_\sigma(u, v), \quad u, v \in \Pi, \quad \gamma \in \Gamma.$$

- (3) *The symplectic \mathbb{F}_p -representation of Γ afforded by Π is anisotropic and faithful.*

Proof. The proposition summarizes [23] Theorem 2.2, Theorem 4.1. \square

We shall refer to the extension T/F as the *imprimitivity field* of σ .

4.3. It will be useful to have an external description of the imprimitivity field T/F arising in 4.2.

Proposition. *Let $\sigma \in \widehat{\mathcal{W}}_F$ be primitive of dimension p^r . Let T/F be the imprimitivity field of σ .*

- (1) *The group $D(\sigma_T)$ has order p^{2r} .*
- (2) *If L/F is a finite tame extension, then $|D(\sigma_L)| \leq p^{2r}$. Moreover, $|D(\sigma_L)| = p^{2r}$ if and only if there exists an F -embedding $T \rightarrow L$.*

In particular, T/F is the unique minimal tame extension for which $D(\sigma_T)$ has order p^{2r} .

Proof. Part (1) has been noted in 4.2, and the first assertion of (2) comes from 4.1 Lemma. Next, let K/F be a finite, Galois, tame extension containing both L and T . By 4.1 Proposition and part (1), the canonical map $D(\sigma_T) \rightarrow D(\sigma_K)$ is bijective while $D(\sigma_L) \rightarrow D(\sigma_K)$ is injective. The latter identifies $D(\sigma_L)$ with the group of $\text{Gal}(K/L)$ -fixed points in $D(\sigma_K)$. However, if $\phi \in D(\sigma_K)$, the $\text{Gal}(K/F)$ -isotropy group of ϕ contains $\text{Gal}(K/T)$, whence the result follows. \square

5. THE LANGLANDS PARAMETER

Let $\pi \in {}^1\mathcal{A}_{p^r}(F)$, for some $r \geq 1$, and set $\sigma = {}^L\pi \in {}^1\mathcal{G}_{p^r}(F)$. We use the representation π to identify the imprimitivity field T/F of σ , and the group $D(\sigma_T)$ of characters χ of \mathcal{W}_T such that $\chi \otimes \sigma_T \cong \sigma_T$. As in 4.2 Lemma, the group of characters $D(\sigma_T)$ determines the p -kernel field E/T of σ .

5.1. We describe π , following the outline of 2.1. We therefore choose a character ψ_F of F such that $c(\psi_F) = -1$. To get clean results, it will be necessary to impose the normalization

$$(5.1.1) \quad \psi_F(\zeta^p) = \psi_F(\zeta), \quad \zeta \in \mu_F,$$

although it will not be invoked until a late stage (5.14 below).

There is an m -simple stratum $[\mathfrak{a}, 1, 0, \alpha]$ in $A = M_{p^r}(F)$, with $e_{\mathfrak{a}} = p^r$, such that π contains the character

$$(5.1.2) \quad \theta_{\alpha} : 1+x \mapsto \psi_A(\alpha x), \quad 1+x \in U_{\mathfrak{a}}^1,$$

where $\psi_A = \psi_F \circ \text{tr}_A$.

Theorem. *The imprimitivity field T of σ is the splitting field over F of the polynomial*

$$(5.1.3) \quad X^{p^{2r}-1} - (-1)^p \det \alpha^{p^r-1}.$$

The non-trivial elements of $D(\sigma_T)$ are the characters Δ_c of T^{\times} such that

$$(5.1.4) \quad \Delta_c^p = 1, \quad \Delta_c|_{\mu_T} = 1, \quad \Delta_c(\det \alpha) = 1,$$

and

$$(5.1.5) \quad \Delta_c(1+y) = \psi_T(cy), \quad 1+y \in U_T^1,$$

where c ranges over the roots of the polynomial (5.1.3) and $\psi_T = \psi_F \circ \text{Tr}_{T/F}$.

5.2. The proof of the theorem will occupy the rest of the section, but first we draw some conclusions.

The conditions (5.1.4), (5.1.5) determine the character Δ_c uniquely. The representation π determines only the coset $(\det \alpha)U_F^1$ (2.2). Changing $\det \alpha$ within that coset changes neither T (as a subfield of \bar{F}) nor the group $D(\sigma_T)$. Let E/F be the p -kernel field of σ . As in 4.2 Lemma, the abelian extension E/T is given by the relation

$$(5.2.1) \quad N_{E/T}(E^\times) = \bigcap_c \text{Ker } \Delta_c,$$

with c ranging over the roots of the polynomial (5.1.3). Thus E is also determined by the coset $(\det \alpha)U_F^1$.

In the other direction, the datum $(T/F, D(\sigma_T))$ does *not* fully determine the coset $(\det \alpha)U_F^1$.

Corollary. *Let $\pi_i \in {}^1\mathcal{A}_{p^r}(F)$ contain the simple stratum $[\mathfrak{a}, 1, 0, \alpha_i]$, $i = 1, 2$. Let E_i/F be the p -kernel field of $\sigma_i = {}^L\pi_i$ and T_i/F its imprimitivity field. The following conditions are equivalent:*

- (1) $E_1 = E_2$;
- (2) $T_1 = T_2$;
- (3) $\det \alpha_1 \equiv \zeta \det \alpha_2 \pmod{U_F^1}$, for some $\zeta \in F^\times$ such that $\zeta^{p^r} = \zeta$.

Proof. Suppose first that $E_1 = E_2$. Therefore $T_1 = T_2$ and comparison of the polynomials (5.1.3) defining the two fields yields condition (3). Supposing that (3) holds, 5.1 Theorem implies $T_1 = T_2 = T$, say, and $D(\sigma_{1,T}) = D(\sigma_{2,T})$. The relation (5.2.1) implies $E_1 = E_2$, as required for (1). \square

5.3. We start the proof of 5.1 Theorem with a preliminary estimate of the imprimitivity field T/F .

Theorem. *Let $\sigma \in {}^1\mathcal{G}_{p^r}(F)$, $r \geq 1$, and let T/F be its imprimitivity field.*

- (1) *The field T satisfies $e(T|F) = 1+p^r$ and it contains a root of unity of order $p^{2r}-1$.*
- (2) *If $\chi \in D(\sigma_T)$ and $\chi \neq 1$, then $\text{sw}(\chi) = 1$.*

Proof. The proof takes until the end of the next sub-section. In this one, we find an upper bound for $e(T|F)$. Set $\Pi = \text{Gal}(E/T)$ and $\Gamma = \text{Gal}(T/F)$.

Proposition. *The $\mathbb{F}_p\Gamma$ -representation afforded by Π is irreducible.*

Proof. Suppose otherwise. From [23] Theorem 12.2 we deduce the existence of $\sigma_1, \sigma_2 \in \widehat{W}_F$ such that $\dim \sigma_i = p^{r_i}$, with $r_2 \geq r_1 \geq 1$, and $\sigma \cong \sigma_1 \otimes \sigma_2$. Let $\pi = {}^L\sigma$, $\pi_i = {}^L\sigma_i$, and let $\varepsilon(\pi_1 \times \pi_2, s, \psi)$ be the local constant of the pair (π_1, π_2) , in the sense of Jacquet, Piatetskii-Shapiro and Shalika [22], Shahidi [29]. This takes the form

$$\varepsilon(\pi_1 \times \pi_2, s, \psi) = q^{-sp^r(\mathfrak{F}(\tilde{\pi}_1, \pi_2) + c(\psi))} \varepsilon(\pi_1 \times \pi_2, 0, \psi),$$

for a certain rational number $\mathfrak{F}(\tilde{\pi}_1, \pi_2)$ worked out in [6], [13]. We have

$$\varepsilon(\sigma, s, \psi) = \varepsilon(\pi_1 \times \pi_2, s, \psi),$$

so $\text{sw}(\sigma) = p^r \mathfrak{F}(\tilde{\pi}_1, \pi_2) - p^r$. In present circumstances, $\tilde{\pi}_1$ cannot be of the form $\chi\pi_2$ for an unramified character χ of F^\times , since that would imply $\sigma_1 \otimes \sigma_2$ reducible. So, from [6] 2.1 Proposition and Corollary, we obtain

$$\mathfrak{F}(\tilde{\pi}_1, \pi_2) > 1 + \frac{\mathfrak{c}}{p^{2r_1}},$$

for a certain integer $\mathfrak{c} \geq 1$. In particular, $\text{sw}(\sigma) > p^{r_2 - r_1} \mathfrak{c} \geq 1$. That is, $\text{sw}(\sigma) > 1$, contrary to hypothesis. \square

Corollary. *The ramification index $e(T|F)$ divides $1+p^r$.*

Proof. Let K/F be the maximal unramified sub-extension of T/F . The representation $\sigma_K = \sigma|_{\mathcal{W}_K}$ satisfies the same hypotheses as σ : it is irreducible, $\dim \sigma_K = p^r$ and $\text{sw}(\sigma_K) = 1$. It is therefore primitive and the proposition applies to it unchanged. In other words, we may assume that the Galois tame extension T/F is *totally ramified*. The cyclic group $\Gamma = \text{Gal}(T/F)$ thus admits an irreducible, faithful, anisotropic, symplectic \mathbb{F}_p -representation of dimension $2r$. By [10] 3.3, $|\Gamma|$ divides $1+p^r$, as required. \square

Remark. The bound $e(T|F) \leq 1+p^r$ applies to any primitive representation of \mathcal{W}_F of dimension p^r , irrespective of the value of the Swan conductor [19]. We have used this more demanding technique for its wider interest.

5.4. We continue the proof of 5.3 Theorem.

Lemma. *Let K/F be a finite tame extension, and set $e_K = e(K|F)$. If there exists a non-trivial character χ of \mathcal{W}_K such that $\chi \otimes \sigma_K \cong \sigma_K$, then $e_K \geq 1+p^r$.*

Proof. The representation σ_K satisfies $\text{sw}(\sigma_K) = e_K$. If χ is a character of \mathcal{W}_K with $\text{sw}(\chi) = s \geq 0$, then $\text{sw}(\chi \otimes \sigma_K) \leq \max\{e_K, p^r s\}$, with equality if $e_K \neq p^r s$ [19] 3.6. A relation $\chi \otimes \sigma_K \cong \sigma_K$ implies first that $s \geq 1$ (cf. 4.3) and second that $p^r s \leq e_K$. Since p does not divide e_K , we get $e_K > p^r$, as required. \square

Applying the lemma to the case $K = T$ and recalling 5.3 Corollary, we get $e_T = e(T|F) = 1+p^r$. Also, if $\chi \in D(\sigma_T)$ is non-trivial and has Swan conductor s , the argument of the preceding proof gives $sp^r < 1+p^r$, whence $s = 1$.

Finally, since T/F is Galois with ramification index $1+p^r$, surely T contains a primitive $1+p^r$ -th root of unity. That is, if $|\mathbb{k}_T| = p^t$, say, then $1+p^r$ divides $p^t - 1$. An elementary argument shows that t is divisible by $2r$, whence follows the only remaining assertion of 5.3 Theorem. \square

5.5. We start the proof of 5.1 Theorem. Let K/F be a finite Galois extension with $e(K|F) = 1+p^r$: in particular, K/F is tamely ramified. The representation σ_K is irreducible, so we define $\pi_K \in \mathcal{A}_{p^r}(K)$ by ${}^L\pi_K = \sigma_K$. A character χ of K^\times then lies in $D(\sigma_K)$ if and only if $\chi\pi_K \cong \pi_K$. We therefore approach 5.1 Theorem via the representation π_K . Following 4.3 Proposition, we have to find the least extension K/F for which the equation $\chi\pi_K \cong \pi_K$ has p^{2r} solutions χ .

We describe π_K in terms of simple strata and simple characters. The field extension $P = F[\alpha]/F$ is totally wildly ramified, of degree p^r . We set $B = \mathcal{M}_{p^r}(K) = K \otimes_F A$. The K -algebra $K \otimes_F P$ is a field. We denote it KP and identify it with a subfield of B . Let \mathfrak{b} be the unique

hereditary \mathfrak{o}_K -order in B which is stable under conjugation by KP^\times . Let \mathfrak{q} be the Jacobson radical of \mathfrak{b} . The K -period $e_{\mathfrak{b}}$ of \mathfrak{b} is $p^r = e(KP|K)$. The quadruple $[\mathfrak{b}, 1+p^r, 0, \alpha]$ is an m -simple stratum in B and

$$H^1(\alpha, \mathfrak{b}) = U_{KP}^1 U_{\mathfrak{b}}^{1+[1+p^r/2]},$$

where $[x]$ denotes the integer part of the real number x .

We put $\psi_K = \psi_F \circ \text{Tr}_{K/F}$. Thus ψ_K is a smooth character of K such that $c(\psi_K) = -1$. It satisfies the analogue of (5.1.1), namely

$$(5.5.1) \quad \psi_K(\zeta^p) = \psi_K(\zeta), \quad \zeta \in \mu_K.$$

Let $\psi_B = \psi_K \circ \text{tr}_B$. We define a simple character $\theta_\alpha^K \in \mathcal{C}(\mathfrak{b}, \alpha, \psi_K)$ by

$$(5.5.2) \quad \begin{aligned} \theta_\alpha^K(1+h) &= \psi_B(\alpha h), & 1+h &\in U_{\mathfrak{b}}^{1+[1+p^r/2]}, \\ \theta_\alpha^K(1+y) &= \theta_\alpha(\text{N}_{KP/P}(1+y)), & 1+y &\in U_{KP}^1. \end{aligned}$$

We remark that the character θ_α is trivial on U_P^1 , so θ_α^K is trivial on U_{KP}^1 .

Proposition. *The representation π_K contains the m -simple character θ_α^K .*

Proof. The character θ_α^K is m -simple, and its endo-equivalence class is the unique K/F -lift of that of θ_α (see [4] for this concept). The proposition follows from [6] Theorem A. \square

5.6. Let ϕ be a character of K^\times , $\phi \neq 1$, such that $\phi\pi_K \cong \pi_K$. Viewing ϕ as a character of \mathcal{W}_K via class field theory, we get $\phi \otimes \sigma_K \cong \sigma_K$. We argue as in 5.4 to conclude that $\text{sw}(\phi) = 1$. Consequently, there is an element $c \in K$, with $v_K(c) = -1$, such that $\phi(1+y) = \psi_K(cy)$, $y \in \mathfrak{p}_K$.

Proposition. *Let $c \in K$, $v_K(c) = -1$.*

- (1) *The quadruple $[\mathfrak{b}, 1+p^r, 0, \alpha+c]$ is an m -simple stratum in B and*

$$H^1(\alpha+c, \mathfrak{b}) = H^1(\alpha, \mathfrak{b}).$$

- (2) *Let ϕ be a character of K^\times such that $\phi(1+y) = \psi_K(cy)$, $y \in \mathfrak{p}_K$. Let Φ denote the character $\phi \circ \det_B|_{H^1(\alpha, \mathfrak{b})}$.*

- (a) *The character $\Phi\theta_\alpha^K$ lies in $\mathcal{C}(\mathfrak{b}, \alpha+c, \psi_K)$ and is contained in $\phi\pi_K$.*
- (b) *If $\phi\pi_K \cong \pi_K$, the simple characters θ_α^K , $\Phi\theta_\alpha^K$ are conjugate in $\text{GL}_{p^r}(K)$.*
- (c) *If the simple characters θ_α^K , $\Phi\theta_\alpha^K$ are conjugate in $\text{GL}_{p^r}(K)$, there is a unique character ϕ' of K^\times that agrees with ϕ on U_K^1 and such that $\phi'\pi_K \cong \pi_K$.*

Proof. Assertion (1) is immediate from the definitions. Part (2)(a) is an instance of [15] Appendix. Part (2)(b) is given by [14] 8.4 (or [11] Corollary 1 for this formulation). In part (2)(c), the totally ramified representations π_K , $\phi\pi_K$ contain the same m -simple character and so $\phi\pi_K \cong \chi\pi_K$, for a tamely ramified character χ of K^\times . The result therefore holds with $\phi' = \chi^{-1}\phi$. The uniqueness of ϕ' is given by 4.1 Proposition. \square

Remark. In the context of part (2)(a) of the proposition, $\mathcal{C}(\mathfrak{b}, \alpha+c, \psi_K) = \Phi\mathcal{C}(\mathfrak{b}, \alpha, \psi_K)$.

5.7. We collect some technical results.

Lemma 1. *If $d(KP|K)$ is the differential exponent of KP/K , then $d(KP|K) \geq 2p^r$.*

Proof. The extension P/F is totally wildly ramified, so $d(P|F) \geq [P:F] = p^r$. The transitivity property of the different gives us first

$$d(KP|F) = d(KP|P) + (1+p^r)d(P|F) \geq p^r + (1+p^r)p^r = 2p^r + p^{2r},$$

and second $d(KP|F) = d(KP|K) + p^{2r}$. It follows that $d(KP|K) \geq 2p^r$, as required. \square

Let $s_K : B \rightarrow KP$ be a tame corestriction on B , relative to KP/K (see [14] (1.3.3) for the definition).

Lemma 2. *For any integer t , we have $s_K(\mathfrak{p}_{KP}^t) \subset \mathfrak{p}_{KP}^{1+p^r+t}$.*

Proof. This follows from Lemma 1 and [14] (1.3.8). \square

5.8. We take $c \in K$ with $v_K(c) = -1$. We compare the \mathfrak{m} -simple strata $[\mathfrak{b}, 1+p^r, 0, \alpha]$, $[\mathfrak{b}, 1+p^r, 0, \alpha+c]$.

Proposition. *There exists $x \in \mathfrak{q}$ such that*

$$(5.8.1) \quad (1+x)^{-1}\alpha(1+x) \equiv \alpha+c \pmod{\mathfrak{b}}.$$

Proof. Let $\mathbb{A} : B \rightarrow B$ be the map $x \mapsto \alpha x \alpha^{-1} - x$. This fits into a long exact sequence [14] 1.4

$$\cdots \rightarrow B \xrightarrow{\mathbb{A}} B \xrightarrow{s_K} B \xrightarrow{\mathbb{A}} B \rightarrow \cdots.$$

Set $\delta = c^{-1}\alpha$: thus δ^{-1} is a prime element of KP and $\delta^{-1}\mathfrak{b} = \mathfrak{q}$. Multiplying the congruence (5.8.1) on the left by $c^{-1}(1+x)$, we see it is equivalent to

$$(5.8.2) \quad \mathbb{A}(x) - x\delta^{-1} \equiv \delta^{-1} \pmod{\mathfrak{q}^{p^r}}.$$

Let i, j be integers, with $1 \leq j \leq p^r$. Since $\mathfrak{p}_K \mathfrak{b} = \mathfrak{q}^{p^r}$, the quotient $V_{i,j} = \mathfrak{q}^i / \mathfrak{q}^{i+j}$ is a \mathbb{k}_K -vector space. Each of the maps s_K, \mathbb{A} induces a \mathbb{k}_K -endomorphism of $V_{i,j}$, for which we use the same notation. Since α is minimal over K , we have an infinite exact sequence [14] (1.4.7), (1.4.15),

$$(5.8.3) \quad \cdots \rightarrow V_{i,j} \xrightarrow{\mathbb{A}} V_{i,j} \xrightarrow{s_K} V_{i,j} \xrightarrow{\mathbb{A}} V_{i,j} \rightarrow \cdots.$$

The subspace $s_K(V_{i,j})$ is the natural image of $\mathfrak{p}_{KP}^i / \mathfrak{p}_{KP}^{i+j}$ in $V_{i,j}$.

Lemma.

(1) *The endomorphism \mathbb{A} of $V_{i,j}$ is nilpotent such that $\mathbb{A}^{p^r} = 0$ and*

$$\mathbb{A}^{p^r-1}(V_{i,j}) = s_K(V_{i,j}) = \mathfrak{p}_{KP}^i / \mathfrak{p}_{KP}^{i+j} \neq 0.$$

(2) *There is a unit $u \in U_{KP}$ such that*

$$\mathbb{A}^{p^r-1}(v) = us_K(v), \quad v \in V_{i,j}.$$

Proof. We consider the relation between the hereditary orders \mathfrak{a} , \mathfrak{b} . Let \mathcal{L} be the chain of \mathfrak{a} -lattices in the vector space $F^{(p^r)}$. There is then a unique \mathfrak{o}_K -lattice chain in $K \otimes F^{(p^r)}$ which is stable under translation by elements of $(K \otimes P)^\times = KP^\times$, and contains every lattice $\mathfrak{o}_K \otimes_{\mathfrak{o}_F} L$, $L \in \mathcal{L}$. The hereditary \mathfrak{o}_K -order defined by this chain is \mathfrak{b} . Consequently, if $\mathfrak{p} = \text{rad } \mathfrak{a}$, then $\mathfrak{p} \subset \mathfrak{q}^{1+p^r}$.

The element α^{p^r} lies in $\varpi_\alpha^{-1}U_{\mathfrak{a}}^1 \subset \varpi_\alpha^{-1}U_{\mathfrak{b}}^{1+p^r}$, for a certain prime element ϖ_α of F . Thus α^{p^r} acts trivially on $V_{i,j}$, and so $\mathbb{A}^{p^r}(V_{i,j}) = 0$. The exact sequence (5.8.3) shows that $\text{Ker } \mathbb{A}$ has dimension $j = p^{-r} \dim V_{i,j}$. We conclude that \mathbb{A} is nilpotent of type (j, j, \dots, j) and all assertions in (1) follow.

It follows that $\text{Ker } \mathbb{A}^{p^r-1} = \text{Im } \mathbb{A} = \text{Ker } s_K$. The maps s_K and \mathbb{A}^{p^r-1} are therefore \mathfrak{o}_{KP} -isomorphisms between two free $\mathfrak{o}_{KP}/\mathfrak{p}_{KP}^j$ -modules of rank one, whence they differ by a unit. \square

We apply the lemma to the case $V = V_{1,p^r-1} = \mathfrak{q}/\mathfrak{q}^{p^r}$. We define another \mathbb{k}_K -endomorphism \mathbb{B}_c of V by

$$\mathbb{B}_c(v) = \mathbb{A}(v) - v\delta^{-1}, \quad v \in V.$$

To prove the proposition, we have to show that the equation $\mathbb{B}_c(x) = \delta^{-1}$ has a solution $x = x_c$ in V . To do this, we abbreviate $m = p^r - 2$ and define

$$X_c(z) = \mathbb{A}^m(z) + \mathbb{A}^{m-1}(z)\delta^{-1} + \mathbb{A}^{m-2}(z)\delta^{-2} + \dots \\ \dots + \mathbb{A}(z)\delta^{1-m} + z\delta^{-m}, \quad z \in V,$$

so that

$$\mathbb{B}_c(X_c(z)) = \mathbb{A}^{m+1}(z) - z\delta^{-(m+1)} = \mathbb{A}^{m+1}(z)$$

in V . By the lemma, we may choose $z \in V$ such that $\mathbb{A}^{m+1}(z) = \mathbb{A}^{p^r-1}(z) = \delta^{-1}$, and then $x = X_c(z)$ provides the desired solution to the congruence (5.8.2). \square

5.9. We gather the threads for the next phase of the proof of 5.1 Theorem.

Let $c \in K$, $v_K(c) = -1$ and let $x = x_c \in \mathfrak{q}$ satisfy

$$(1+x)^{-1}\alpha(1+x) \equiv \alpha+c \pmod{\mathfrak{b}},$$

as in 5.8 Proposition. The formula $\psi_{K,c} : 1+t \mapsto \psi_K(ct)$ defines a character of U_K^1 . This gives a character

$$\chi_c : h \mapsto \psi_{K,c}(\det_B h)$$

of $H^1(\alpha, \mathfrak{b})$. As in 5.6 Remark, $\mathcal{C}(\mathfrak{b}, \alpha+c, \psi_K) = \mathcal{C}(\mathfrak{b}, \alpha, \psi_K) \chi_c$.

Lemma. *The element $1+x$ normalizes $H^1(\alpha, \mathfrak{b}) = H^1(\alpha+c, \mathfrak{b})$ and conjugation by $1+x$ gives a bijection*

$$\mathcal{C}(\mathfrak{b}, \alpha, \psi_K) \longrightarrow \mathcal{C}(\mathfrak{b}, \alpha+c, \psi_K), \\ \vartheta \longmapsto \vartheta^{1+x}.$$

There exists $\vartheta_c \in \mathcal{C}(\mathfrak{b}, \alpha, \psi_K)$ such that $(\theta_\alpha^K)^{1+x} = \vartheta_c \chi_c$.

Proof. The equality of H^1 -groups has been noted in 5.6 Proposition, whence the other assertions follow. \square

The next step in the proof of 5.1 Theorem is to find all elements c for which $\vartheta_c = \theta_\alpha^K$, that is,

$$(5.9.1) \quad (\theta_\alpha^K)^{1+x_c} = \theta_\alpha^K \chi_c.$$

For such an element c , 5.6 Proposition gives a character Δ_c of K^\times , extending $\psi_{K,c}$, such that $\Delta_c \pi_K \cong \pi_K$.

5.10. We have to compute the quantity

$$\theta_\alpha^K((1+x)h(1+x)^{-1}), \quad h \in H^1(\alpha, \mathfrak{b}) = U_{KP}^1 U_{\mathfrak{b}}^{1+[1+p^r/2]},$$

where $x = x_c$. The definition of x_c yields

$$(5.10.1) \quad \theta_\alpha^K((1+x)h(1+x)^{-1}) = \theta_\alpha^K(h) \chi_c(h), \quad h \in U_{\mathfrak{b}}^{1+[1+p^r/2]}.$$

It is therefore enough to consider elements $h = 1+y \in U_{KP}^1$.

Proposition. *If $y \in \mathfrak{p}_{KP}$ and $x = x_c$, then $(\theta_\alpha^K)^{1+x}(1+y) = \psi_B(-cxy)$.*

Proof. We need a preliminary calculation.

Lemma. *If $y \in \mathfrak{p}_{KP}$, then*

$$(1+x)(1+y)(1+x)^{-1} \equiv 1+v \pmod{U_{\mathfrak{b}}^{1+p^r}},$$

where $v \in \mathfrak{p}_{KP}$ and $v \equiv y \pmod{U_{KP}^2}$.

Proof. We write $[x, y] = xy - yx$, so that

$$1+t = (1+x)(1+y)(1+x)^{-1} = 1+y + [x, y](1+x)^{-1}.$$

Surely $t \equiv y \pmod{\mathfrak{q}^2}$, so the result reduces to showing $\alpha t - t\alpha \in \alpha \mathfrak{q}^{1+p^r} = \mathfrak{b}$. We use two forms of the defining relation for x , namely

$$\begin{aligned} \alpha x - x\alpha &\equiv c(1+x) \pmod{\mathfrak{b}}, \\ (1+x)^{-1}\alpha &\equiv (\alpha+c)(1+x)^{-1} \pmod{\mathfrak{b}}. \end{aligned}$$

Expanding,

$$\begin{aligned} \alpha t - t\alpha &= \alpha[x, y](1+x)^{-1} - [x, y](1+x)^{-1}\alpha \\ &= (\alpha[x, y] - [x, y](\alpha+c))(1+x)^{-1} \\ &\equiv \alpha[x, y] - [x, y](\alpha+c) \pmod{\mathfrak{b}}. \end{aligned}$$

We recall that c is central and that y commutes with α . Expanding the commutators, the defining relation yields $\alpha t - t\alpha \equiv 0 \pmod{\mathfrak{b}}$, as required. \square

We use the lemma to write

$$(1+x)(1+y)(1+x)^{-1} = 1+v+h, \quad h \in \mathfrak{q}^{1+p^r}, \quad v \in \mathfrak{p}_{KP},$$

with $v \equiv y \pmod{\mathfrak{p}_{KP}^2}$. Thus

$$(\theta_\alpha^K)^{1+x}(1+y) = \theta_\alpha^K(1+y) \psi_B(\alpha h).$$

We have $\theta_\alpha^K(1+y) = 1 = \psi_B(\alpha y)$ by (5.5.2), 5.7 Lemma 2, respectively, so

$$(\theta_\alpha^K)^{1+x}(1+y) = \psi_B(\alpha(v+h)), \quad \text{and} \quad v+h = y + [x, y](1+x)^{-1}.$$

We have to compute

$$\psi_B(\alpha[x, y](1+x)^{-1}) = \psi_B\left(\sum_{i \geq 0} (-1)^i \alpha(xy-yx)x^i\right).$$

We expand the inner sum using the symmetry properties of the trace and the defining relation for x in the form

$$[\alpha, x] \equiv c(1+x) \pmod{\mathfrak{b}}.$$

The term $i = 0$ contributes $\psi_B(\alpha(xy-yx)) = 1$, since y commutes with α . The general term $i \geq 1$ gives

$$\psi_B((-1)^i \alpha(xy-yx)x^i) = \psi_B((-1)^i x^i [\alpha, x]y) = \psi_B((-1)^i c(x^i + x^{i+1})).$$

In all, we get

$$\psi_B(\alpha[x, y](1+x)^{-1}) = \psi_B(-cxy),$$

or $(\theta_\alpha^K)^{1+x}(1+y) = \psi_B(-cxy)$, as required. \square

5.11. We examine the character

$$\eta_c : 1+y \mapsto \psi_B(-cxy)$$

of U_{KP}^1 arising in 5.10 Proposition. We use the map \mathbb{A} of 5.8 and let s_α^K be a tame corestriction on B , relative to KP/K , such that

$$s_\alpha^K(t) \equiv \mathbb{A}^{p^r-1}(t) \pmod{\mathfrak{q}^{p^r}}, \quad t \in \mathfrak{q},$$

(cf. 5.8 Lemma (2)).

Lemma. *There is a unique character ϵ of KP such that $c(\epsilon) = -1$ and*

$$(5.11.1) \quad \psi_B(b) = \epsilon(s_\alpha^K(b)), \quad b \in B.$$

The character η_c then satisfies

$$(5.11.2) \quad \eta_c(1+y) = \epsilon(-\alpha^{1-p^r} c^{p^r} y), \quad y \in \mathfrak{p}_{KP}.$$

Consequently, η_c is trivial on U_{KP}^2 .

Proof. The first assertion is given by [14] (1.3.7).

We return to the construction of x_c in 5.8. As there, we put $\delta = c^{-1}\alpha$. We choose $z \in \mathfrak{q}$ such that $\mathbb{A}^{p^r-1}(z) \equiv \delta^{-1} \pmod{\mathfrak{q}^{p^r}}$: this yields $x_c = X_c(z)$. The choice of s_α^K then gives

$$s_\alpha^K(cx_c) \equiv c\delta^{1-p^r} \pmod{\mathfrak{b}},$$

while $c\delta^{1-p^r} = \alpha^{1-p^r} c^{p^r}$. The lemma now follows from the relation (5.11.1). \square

5.12. We compare the characters η_c, χ_c on U_{KP}^1 . Both are trivial on U_{KP}^2 and, for $y \in \mathfrak{p}_{KP}$,

$$\chi_c(1+y) = \psi_{K,c}(\mathrm{N}_{KP/K}(1+y)).$$

We may take $y = \zeta c \alpha^{-1}$, for some $\zeta \in \mu_{KP} = \mu_K$.

Lemma. *If $y = \zeta c \alpha^{-1}$, $\zeta \in \mu_K$, then*

$$\mathrm{N}_{KP/K}(1+y) \equiv 1 + \zeta^{p^r} c^{p^r} \det \alpha^{-1} \pmod{U_K^2}.$$

Proof. Nothing is changed if we replace α by αu , $u \in U_a^1$. We may therefore assume (as in the proof of 2.2 Proposition) that α^{-p^r} is a prime element of F . Therefore $(c\alpha^{-1})^{p^r}$ is a prime element of K . We may choose a matrix representation so that $c\alpha^{-1}$ is a monomial matrix as in 2.2. We may re-write the identity to be proved as

$$\det(1+y) \equiv 1 + \det(\zeta c \alpha^{-1}) \pmod{U_K^2}.$$

The lemma is then given by an elementary calculation. \square

It follows that

$$(5.12.1) \quad \chi_c(1+y) = \psi_K(\zeta^{p^r} c^{1+p^r} \det \alpha^{-1}), \quad y = \zeta c \alpha^{-1}.$$

In these terms, (5.11.2) says

$$(5.12.2) \quad \eta_c(1+y) = \epsilon((-1)^{p^r} c^{1+p^r} \zeta \det \alpha^{-1}), \quad y = \zeta c \alpha^{-1}.$$

5.13. We relate the characters ψ_K, ϵ .

Lemma. *The characters ϵ, ψ_K satisfy $\epsilon(\zeta) = \psi_K(\zeta)$, for all $\zeta \in \mu_K$.*

Proof. We continue in the situation of the proof of 5.12 Lemma. For $\zeta \in \mu_K$, we let ζ_0 be the matrix with ζ in the $(1,1)$ -place and with all other entries zero. The matrix $\mathbb{A}^{p^r-1}(\zeta_0)$ is a scalar matrix whose last entry is ζ , whence the lemma follows. \square

5.14. Let $\varsigma_c \in \mu_K$ satisfy $\varsigma_c \equiv c^{1+p^r} \det \alpha^{-1} \pmod{U_{KP}^1}$. Combining (5.12.1), (5.12.2), we have to solve the equation

$$\psi_K((-1)^{1+p^r} \varsigma_c \zeta) = \psi_K(\varsigma_c \zeta^{p^r}), \quad \zeta \in \mu_K,$$

for $c \in K$, $v_K(c) = -1$. We have $\psi_K(\gamma^p) = \psi_K(\gamma)$, for $\gamma \in \mu_K$ (5.5.1). We must therefore solve $(-1)^{p^{2r}} \varsigma_c^{p^r} = \varsigma_c$, that is, $\varsigma_c^{1-p^r} = (-1)^p$. This translates back to

$$c^{1-p^{2r}} \equiv (-1)^p \det \alpha^{1-p^r} \pmod{U_K^1}.$$

This congruence admits $p^{2r}-1$ solutions in K^\times/U_K^1 if and only if the field K contains the splitting field of the polynomial (5.1.3).

Let K be this splitting field and let $c \in K$ be a root of the polynomial (5.1.3). Let ϕ_c be a character of K^\times such that $\phi_c(1+t) = \psi_K(ct)$, $t \in \mathfrak{p}_K$. The representations $\pi_K, \phi_c \pi_K$ both contain the simple character θ_α^K . As in 5.6 Proposition, there is a unique character Δ_c of K^\times , agreeing with ϕ_c on U_K^1 , such that $\Delta_c \pi_K \cong \pi_K$. That is, $\Delta_c \in D(\pi_K)$. Surely K is minimal for the property $|D(\sigma_K)| = p^{2r}$, so $K = T$, as desired.

This character Δ_c satisfies (5.1.5) by construction. By (5.1.5), Δ_c^p is tamely ramified and so trivial, by 4.1 Proposition. The second property in (5.1.4) expresses the fact that $\pi_K, \Delta_c \pi_K$ have the same central character, while the third follows from the relation $\varepsilon(\Delta_c \pi_K, s, \psi_T) = \varepsilon(\pi_K, s, \psi_T)$.

We have completed the proof of 5.1 Theorem. \square

6. THE P-CENTRAL CHARACTER

As in §5, $\pi \in {}^1\mathcal{A}_{p^r}(F)$ contains the simple stratum $[\mathfrak{a}, 1, 0, \alpha]$, relative to the character ψ_F of (5.1.1). From 5.1 Theorem, the primitive representation $\sigma = {}^L\pi$ has p -kernel field E/F and imprimitivity field T/F . We investigate the p -central character ξ_σ of σ to complete our examination of the representation σ .

6.1. Let F'/F be the maximal unramified sub-extension of T/F . Thus T/F' is cyclic and totally ramified of degree $1+p^r$. We set

$$(6.1.1) \quad \begin{aligned} \Pi &= \text{Gal}(E/T) \cong (\mathbb{Z}/p\mathbb{Z})^{2r}, \\ \Gamma &= \text{Gal}(T/F') \cong \mathbb{Z}/(1+p^r)\mathbb{Z}, \\ \Theta &= \text{Gal}(E/F'). \end{aligned}$$

The extension E/T is totally wildly ramified, so there exists $\beta_E \in E^\times$ such that $N_{E/T}(\beta_E) \equiv \det \alpha \pmod{U_T^1}$. This condition determines the coset $\beta_E U_E^1$ uniquely. We write $\psi_E = \psi_F \circ \text{Tr}_{E/F}$.

Theorem. *The character ξ_σ has the following properties:*

- (1) ξ_σ is fixed under conjugation by Θ ,
- (2) $\text{sw}(\xi_\sigma) = 1+p^r$, and
- (3) if $\beta_E \in E$ satisfies $N_{E/T}(\beta_E) \equiv \det \alpha \pmod{U_T^1}$, then

$$\xi_\sigma(1+x) = \psi_E(\beta_E^{p^r} x), \quad 1+x \in U_E^{1+p^r}.$$

The properties (1)–(3) determine the character $\xi_\sigma|_{U_E^1}$ uniquely.

We give a more complete formula for $\xi_\sigma|_{U_E^1}$ in (6.7.2) below. We note the following refinement of 5.2 Corollary.

Corollary. *Let $\pi_i \in {}^1\mathcal{A}_{p^r}(F)$ contain the m -simple stratum $[\mathfrak{a}, 1, 0, \alpha_i]$, $i = 1, 2$. Suppose that the representations $\sigma_i = {}^L\pi_i$ have p -kernel field E . If ξ_i is the p -central character of σ_i , the following conditions are equivalent.*

- (1) $\xi_1|_{U_E^1} = \xi_2|_{U_E^1}$;
- (2) $\xi_1|_{U_E^{1+p^r}} = \xi_2|_{U_E^{1+p^r}}$;
- (3) $\det \alpha_1 \equiv \det \alpha_2 \pmod{U_F^1}$.

The proofs occupy the rest of the section.

6.2. Before starting, we show how 6.1 Theorem completes our picture of the representation σ . We use the viewpoint of [12] §1. According to the Ramification Theorem of local class field theory, the Artin Reciprocity map $\mathcal{W}_E \rightarrow E^\times$ maps the wild inertia subgroup \mathcal{P}_E onto U_E^1 . From this point of view, the restricted character $\xi_\sigma|_{U_E^1}$, determined in 6.1 Theorem, gives a character ζ_σ of \mathcal{P}_E . (If we think of ξ_σ as a character of \mathcal{W}_E , then $\zeta_\sigma = \xi_\sigma|_{\mathcal{P}_E}$.)

Corollary.

- (1) *The representation $\rho_\sigma = \sigma|_{\mathcal{P}_F}$ is irreducible. It is the unique irreducible representation of \mathcal{P}_F containing ζ_σ .*

(2) The representation σ is the unique element of ${}^1\mathcal{G}_{p^r}(F)$ with the following properties:

- (a) $\sigma|_{\mathcal{P}_F} \cong \rho_\sigma$,
- (b) $\det \sigma = \omega_\pi$, and
- (c) $\varepsilon(\sigma, \frac{1}{2}, \psi_F) = \varepsilon(\pi, \frac{1}{2}, \psi_F)$.

Proof. The representation ρ_σ is surely irreducible and contains ζ_σ . The relation $\text{Ind}_{\mathcal{W}_E}^{\mathcal{W}_T} \xi_\sigma = p^r \sigma_T$ of 4.2 Lemma implies $\text{Ind}_{\mathcal{P}_E}^{\mathcal{P}_F} \zeta_\sigma = p^r \rho_\sigma$, whence (1) follows.

That σ has the listed properties follows from the definition of ρ_σ and 2.3 Proposition. If σ' is some other representation of \mathcal{W}_F extending ρ_σ , 1.3 Proposition of [12] asserts that $\sigma' \cong \chi \otimes \sigma$, for a unique tamely ramified character χ of \mathcal{W}_F . If $\det \sigma' = \chi^{p^r} \det \sigma = \det \sigma$, then $\chi^{p^r} = 1$ and consequently χ is unramified. So, by 2.3 Lemma and Proposition,

$$\varepsilon(\sigma', \frac{1}{2}, \psi_F) = \chi(\det \alpha)^{-1} \varepsilon(\sigma, \frac{1}{2}, \psi_F).$$

Since $v_F(\det \alpha) = -1$, a relation $\varepsilon(\sigma', \frac{1}{2}, \psi_F) = \varepsilon(\sigma, \frac{1}{2}, \psi_F)$ implies $\chi = 1$. \square

Remark. The field extension E/F and the representation ρ_σ of \mathcal{P}_F are determined entirely by the coset $(\det \alpha)U_F^1$ or, equivalently, by the endo-class Θ_α of the simple character θ_α occurring in π (cf. 2.1). Indeed, Θ_α is the endo-class corresponding to ρ_σ via the Ramification Theorem of [12] 6.1.

6.3. We use the notation set up at start of 6.1.

A subgroup Ξ of Π is ξ_σ -Lagrangian if $|\Xi| = p^r$ and the commutator pairing $(x, y) \mapsto \xi_\sigma[x, y]$ is null on Ξ . Equivalently, Ξ is the image in Π of a maximal abelian subgroup $\tilde{\Xi}$ of $\sigma(\mathcal{W}_T)$. It follows that ξ_σ extends to a character of $\tilde{\Xi}$ or, in terms of fields, ξ_σ factors through the norm map $N_{E/K} : E^\times \rightarrow K^\times$ where $K = E^\Xi$, cf. 4.2 Lemma. We identify the dual $\widehat{\Xi}$ of the abelian group Ξ with the group of characters of K^\times vanishing on norms from E .

Proposition. Let Ξ be an ξ_σ -Lagrangian subgroup of Π and set $K = E^\Xi$.

(1) The fields $T \subset K \subset E$ satisfy

$$\begin{aligned} d(E|T) &= 2p^{2r} - 2, \\ d(E|K) &= d(K|T) = 2p^r - 2. \end{aligned}$$

(2) If $\phi \in \widehat{\Xi}$, $\phi \neq 1$, then $\text{sw}(\phi) = 1$.

(3) The character ξ_σ satisfies $\text{sw}(\xi_\sigma) = 1 + p^r$.

(4) If χ is a character of K^\times , such that $\xi_\sigma = \chi \circ N_{E/K}$, then $\text{sw}(\chi) = 2$.

Proof. Each of the extensions E/K , K/T is totally ramified and $\Xi \cong (\mathbb{Z}/p\mathbb{Z})^r \cong \text{Gal}(K/T)$. As in 5.1, every non-trivial $\phi \in D(\sigma_T) = \widehat{\Pi}$ satisfies $\text{sw}(\phi) = 1$, so the Artin conductor $a(\phi)$ is 2. By the conductor-discriminant formula [28] VI §3 Corollaire 2,

$$d(E|T) = \sum_{\phi \in D(\sigma_T)} a(\phi) = 2p^{2r} - 2.$$

Let $\Psi = \text{Gal}(K/T)$. The dual $\widehat{\Psi}$ of Ψ is a subgroup of $D(\sigma_T) = \widehat{\Pi}$ of order p^r . The relation $d(K|T) = 2p^r - 2$ follows as before. Multiplicativity of the different in towers yields the final part of (1). Part (2) now follows from the conductor-discriminant formula.

In part (3), we recall that $\text{Ind}_{E/T} \xi_\sigma = p^r \sigma_T$. Since $\text{sw}(\sigma_T) = 1 + p^r$, the result is given by part (1) and (1.3.1). Finally, $\text{Ind}_{K/T} \chi = \sigma_T$ (4.2 Lemma) and part (4) follows similarly. \square

6.4. We take Ξ and $K = E^\Xi$ as before.

Proposition. *The norm map $N_{E/K}$ induces an isomorphism*

$$\begin{aligned} U_E^{1+p^r}/U_E^{2+p^r} &\cong U_K^2/U_K^3, \\ N_{E/K}(1+x) &\equiv 1 + \text{Tr}_{E/K}(x) \pmod{U_K^3}, \end{aligned}$$

and an exact sequence

$$1 \rightarrow \Xi \longrightarrow U_E^1/U_E^2 \xrightarrow{N_{E/K}} U_K^1/U_K^2.$$

Proof. We consider the ramification subgroups Ξ_i , $i \in \mathbb{Z}$, $i \geq 0$, of $\Xi = \text{Gal}(E/K)$ in the lower numbering. We recall from [28] IV §2 Proposition 4 that

$$2p^r - 2 = d(E|K) = \sum_{i \geq 0} (|\Xi_i| - 1).$$

Here we have $\Xi = \Xi_0 = \Xi_1$ and $|\Xi| = p^r$. The only conclusion is that Ξ_2 is trivial. The derivative of the Herbrand function $\varphi_{E/K}$ is therefore given by

$$\varphi'_{E/K}(x) = \begin{cases} 1, & 0 < x < 1, \\ p^{-r}, & 1 < x, \end{cases}$$

so the inverse Herbrand function $\Psi_{E/K}$ satisfies

$$\Psi'_{E/K}(x) = \begin{cases} 1, & 0 < x < 1, \\ p^r, & 1 < x. \end{cases}$$

Now we use [28] V Proposition 9. Abbreviating $\Psi = \Psi_{E/K}$, we have an exact sequence

$$(6.4.1) \quad 1 \rightarrow \Xi_{\Psi(i)}/\Xi_{\Psi(i)+1} \longrightarrow U_E^{\Psi(i)}/U_E^{\Psi(i)+1} \xrightarrow{N_{E/K}} U_K^i/U_K^{i+1},$$

for any integer $i \geq 1$.

Since $\Psi(2) = 1+p^r$, the norm $N_{E/K}$ induces an isomorphism $U_E^{1+p^r}/U_E^{2+p^r} \cong U_K^2/U_K^3$. Since $d(E|K) = 2p^r - 2$ (6.3), the trace $\text{Tr}_{E/K}$ induces an isomorphism $\mathfrak{p}_E^{1+p^r}/\mathfrak{p}_E^{2+p^r} \rightarrow \mathfrak{p}_K^2/\mathfrak{p}_K^3$. On the other hand, for $x \in \mathfrak{p}_E^{1+p^r}$,

$$N_{E/K}(1+x) = 1 + \text{Tr}_{E/K}(x) + R,$$

where the remainder term R lies in $\mathfrak{p}_E^{2+2p^r} \cap K \subset \mathfrak{p}_K^3$.

In the second assertion, the exact sequence is the case $i = 1$ of (6.4.1). \square

6.5. Again let Ξ be an ξ_σ -Lagrangian subgroup of Π and $K = E^\Xi$.

Proposition. *Let χ be a character of K^\times such that $\xi_\sigma = \chi \circ N_{E/K}$. Let $\psi_K = \psi_F \circ \text{Tr}_{K/F}$ and let $\beta_K \in K^\times$. The following conditions are equivalent:*

- (1) $\chi(1+y) = \psi_K(\beta_K y)$, $y \in \mathfrak{p}_K^2$;
- (2) $N_{K/T} \beta_K \equiv \det \alpha \pmod{U_T^1}$.

Proof. If $\psi_T = \psi_F \circ \text{Tr}_{T/F}$, then $c(\psi_T) = -1$ since T/F is tame. It follows from 6.3 Proposition and (1.3.1) that

$$c(\psi_K) = d(K|T) + e(K|T)c(\psi_T) = p^r - 2.$$

Let $\beta_K \in K^\times$ satisfy (1). The coset $\beta_K U_K^1$ is thereby uniquely determined, and $v_K(\beta_K) = -(1+p^r)$.

Let $X_1(T)$ denote the group of tamely ramified characters of T^\times . Let $\epsilon \in X_1(T)$ and set $\epsilon_K = \epsilon \circ N_{K/T}$. The map $\epsilon \mapsto \epsilon_K$ is an isomorphism $X_1(T) \rightarrow X_1(K)$, since K/T is totally wildly ramified. The induction relation

$$\text{Ind}_{K/T} \epsilon_K \otimes \chi = \epsilon \otimes \sigma_T,$$

implies the local constant relation

$$\frac{\varepsilon(\epsilon \otimes \sigma_T, s, \psi_T)}{\varepsilon(\sigma_T, s, \psi_T)} = \frac{\varepsilon(\epsilon_K \otimes \chi, s, \psi_K)}{\varepsilon(\chi, s, \psi_K)}.$$

By 5.5 Proposition and a calculation parallel to 2.2 Lemma, the element γ_{σ_T} of 2.3 Lemma is $\det \alpha$ (modulo U_T^1). It follows that $\epsilon(\det \alpha) = \epsilon_K(\beta_K)$ for all $\epsilon \in X_1(T)$, and therefore $\det \alpha \equiv N_{K/T}(\beta_K) \pmod{U_T^1}$.

Conversely, let $\beta'_K \in K^\times$ satisfy $N_{K/T} \beta'_K \equiv \det \alpha \pmod{U_T^1}$. It follows that $\beta'_K \equiv \beta_K \pmod{U_K^1}$, so $\psi_K(\beta'_K y) = \psi_K(\beta_K y) = \chi(1+y)$, $y \in \mathfrak{p}_K^2$. \square

We re-formulate the proposition to avoid the choice of a Lagrangian.

Corollary. *There exists $\beta_E \in E^\times$ such that $N_{E/T} \beta_E \in (\det \alpha) U_T^1$. For any such element β_E , we have*

$$\xi_\sigma(1+z) = \psi_E(\beta_E^{p^r} z), \quad z \in \mathfrak{p}_E^{1+p^r},$$

where $\psi_E = \psi_F \circ \text{Tr}_{E/F}$.

Proof. The first assertion is immediate. If $\beta_K = N_{E/K} \beta_E$, then $\det \alpha \equiv N_{K/T} \beta_K \pmod{U_T^1}$ while $\beta_K \equiv \beta_E^{p^r} \pmod{U_E^1}$. By 6.4 Proposition,

$$N_{E/K}(1+z) \equiv 1 + \text{Tr}_{E/K}(z) \pmod{\mathfrak{p}_K^3}, \quad z \in \mathfrak{p}_E^{1+p^r},$$

so $\xi_\sigma(1+z) = \psi_K(\beta_K \text{Tr}_{E/K}(z)) = \psi_E(\beta_K z) = \psi_E(\beta_E^{p^r} z)$, as required. \square

6.6. By definition, the character ξ_σ is fixed under the action of $\Theta = \text{Gal}(E/F')$ and, by 6.3 Proposition, $\text{sw}(\xi_\sigma) = 1+p^r$. We examine some implications of these properties.

The soluble group Θ has order $p^{2r}(1+p^r)$. The Hall Subgroup Theorem gives a subgroup Δ of Θ of order $1+p^r$, unique up to conjugation in Θ . If we set $H = E^\Delta$, the extension E/H is totally tamely ramified of degree $1+p^r$ and so Δ is cyclic.

Proposition. *Let Δ be a subgroup of Θ of order $1+p^r$, and let $H = E^\Delta$.*

- (1) *Let τ be a character of E^\times fixed by Δ . There exists a character τ^H of H^\times such that $\tau = \tau^H \circ N_{E/H}$. Moreover, $\text{sw}(\tau) = (1+p^r) \text{sw}(\tau^H)$.*

(2) The norm map $N_{E/H}$ induces an isomorphism

$$\begin{aligned} U_E^{1+p^r}/U_E^{2+p^r} &\longrightarrow U_H^1/U_H^2, \\ 1+x &\longmapsto 1 + \text{Tr}_{E/H}(x). \end{aligned}$$

(3) Let τ_1, τ_2 be Δ -fixed characters of E^\times satisfying $\text{sw}(\tau_1) = \text{sw}(\tau_2) = 1+p^r$. If τ_1, τ_2 agree on $U_E^{1+p^r}$, then $\tau_2 = \phi\tau_1$, for a tamely ramified character ϕ of E^\times .

Proof. Parts (1) and (2) are standard properties of cyclic, tamely ramified extensions. Part (3) follows from (1) and (2). \square

6.7. We interpolate an alternative description of the character $\xi_\sigma|_{U_E^1}$. Taking Δ and $H = E^\Delta$ as in 6.6, choose $\beta_H \in H$ so that

$$(6.7.1) \quad N_{H/F'} \beta_H \equiv \det \alpha \pmod{U_{F'}^1}.$$

The relation (6.7.1) determines β_H modulo U_H^1 , that is, modulo $U_E^{1+p^r}$.

Let $\psi_H = \psi_F \circ \text{Tr}_{H/F}$, and define a character ξ^H of U_H^1/U_H^2 by

$$\xi^H(1+x) = \psi_H(\beta_H^{p^r} x), \quad x \in \mathfrak{p}_H.$$

Thus $\xi^H \circ N_{E/H}$ is a character of U_E^1 , fixed by Δ and agreeing with ξ_σ on $U_E^{1+p^r}$. That is,

$$(6.7.2) \quad \xi_\sigma(u) = \xi^H(N_{E/H}(u)), \quad u \in U_E^1,$$

by 6.6 Proposition and 6.5 Corollary.

6.8. We prove 6.1 Theorem. Property (1) is a direct consequence of the definition of ξ_σ , while (2) is given by 6.3 Proposition. We have just established (3) in 6.5 Corollary. The uniqueness assertion is given by 6.6 Proposition. \square

We prove 6.1 Corollary. The equivalence of (1) and (2) is given by 6.5 Proposition, and that of (2) and (3) by 6.5 Corollary and 5.2 Corollary. \square

Remark. Theorem 6.1 gives ξ_σ on U_E^1 while $\xi_\sigma^{p^r} = \det \sigma|_{\mathcal{W}_E} = \omega_\pi \circ N_{E/F}$. Together, these relations determine ξ_σ up to an unramified factor of order dividing p^r . One cannot isolate this factor via the standard technique of a local constant calculation. The character $\psi_E = \psi_F \circ \text{Tr}_{E/F}$ has $c(\psi_E) = p^{2r} - 2$ (by (1.3.2) and 6.3 Proposition). So, if χ is unramified and $\chi^{p^r} = 1$, then $\varepsilon(\chi\xi_\sigma, s, \psi_E) = \varepsilon(\xi_\sigma, s, \psi_E)$.

REFERENCES

1. C.J. Bushnell, *Hereditary orders, Gauss sums and supercuspidal representations of GL_N* , J. reine angew. Math. **375/376** (1987), 184–210.
2. C.J. Bushnell and A. Fröhlich, *Gauss sums and p -adic division algebras*, Lecture Notes in Math., vol. 987, Springer, 1983.
3. ———, *Non-abelian congruence Gauss sums and p -adic simple algebras*, Proc. London Math. Soc. (3) **50** (1985), 207–264.
4. C.J. Bushnell and G. Henniart, *Local tame lifting for $GL(N)$ I: simple characters*, Publ. Math. IHES **83** (1996), 105–233.
5. ———, *Local tame lifting for $GL(n)$ II: wildly ramified supercuspidals*, Astérisque **254** (1999), 1–105.

6. ———, *Local tame lifting for $GL(n)$ IV: simple characters and base change*, Proc. London Math. Soc. (3) **87** (2003), 337–362.
7. ———, *The essentially tame local Langlands correspondence, I*, J. Amer. Math. Soc. **18** (2005), 685–710.
8. ———, *The essentially tame local Langlands correspondence, II: totally ramified representations*, Compositio Mathematica **141** (2005), 979–1011.
9. ———, *The local Langlands Conjecture for $GL(2)$* , Grundlehren der mathematischen Wissenschaften **335**, Springer, 2006.
10. ———, *The essentially tame local Langlands correspondence, III: the general case*, Proc. London Math. Soc. (3) **101** (2010), 497–553.
11. ———, *Intertwining of simple characters in $GL(n)$* , Internat. Math. Res. Notices; doi:10.1093/imrn/rns162 (2012).
12. *To an effective local Langlands Correspondence*, Memoirs Amer. Math. Soc., to appear. arXiv:1103.5316 (2011).
13. C.J. Bushnell, G.M. Henniart and P.C. Kutzko, *Local Rankin-Selberg convolutions for GL_n : explicit conductor formula*, J. Amer. Math. Soc. **11** (1998), 703–730.
14. C.J. Bushnell and P.C. Kutzko, *The admissible dual of $GL(N)$ via compact open subgroups*, Annals of Math. Studies, vol. 129, Princeton University Press, 1993.
15. ———, *The admissible dual of $SL(N)$ II*, Proc. London Math. Soc. (3) **68** (1992), 317–379.
16. P. Deligne and G. Henniart, *Sur la variation, par torsion, des constantes locales d'équations fonctionnelles des fonctions L* , Invent. Math. **64** (1981), 89–118.
17. R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Math., vol. 260, Springer, 1972.
18. B.H. Gross and M. Reeder, *Arithmetic invariants of discrete Langlands parameters*, Duke Math. J. **154** (2010), 431–508.
19. G. Henniart, *Représentations du groupe de Weil d'un corps local*, L'Enseign. Math. **26** (1980), 155–172.
20. ———, *La conjecture de Langlands locale pour $GL(3)$* , Memoire Soc. Math. France, vol. 11/12, Gauthiers-Villars, 1984.
21. H. Jacquet, *Principal L -functions of the linear group*, Automorphic forms, representations and L -functions (A. Borel and W. Casselman, eds.), Proc. Symposia Pure Math., vol. XXXIII part 2, Amer. Math. Soc., 1979, pp. 63–86.
22. H. Jacquet, I. I. Piatetskii-Shapiro and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), 367–483.
23. H. Koch, *Classification of the primitive representations of the Galois group of local fields*, Invent. Math. **40** (1977), 195–216.
24. P.C. Kutzko, *The Langlands conjecture for GL_2 of a local field*, Ann. Math. **112** (1980), 381–412.
25. C. Mœglin, *Sur la correspondance de Langlands-Kazhdan*, J. Math. Pures et Appl. (9) **69** (1990), 175–226.
26. M. Reeder and J.-K. Yu, *Epipelagic representations and invariant theory*, J. Amer. Math. Soc., to appear.
27. J. F. Rigby, *Primitive linear groups containing a normal nilpotent subgroup larger than the centre of the group*, J. London Math. Soc. **35** (1960), 389–400.
28. J.-P. Serre, *Corps locaux*, Hermann, Paris, 1968.
29. F. Shahidi, *Fourier transforms of intertwining operators and Plancherel measures for $GL(n)$* , Amer. J. Math. **106** (1984), 67–111.
30. J. Tate, *Number theoretic background*, Automorphic forms, representations and L -functions (A. Borel and W. Casselman, eds.), Proc. Symposia Pure Math., vol. XXXIII part 2, Amer. Math. Soc., 1979, pp. 3–26.

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